Stats 116 Linear Algebra Section

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Question S.1: Recall that vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ are *linearly dependent* if for some $i \in \{1, \ldots, m\}$, there exist scalars $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that

$$x_i = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_{i-1} x_{i-1} + \beta_{i+1} x_{i+1} + \cdots + \beta_m x_m.$$

The collection $\{x_1, \ldots, x_m\}$ is *linearly independent* if there is no such index i and collection of scalars β_j. Recall also that the *span* of a set of vectors is the set of linear combinations of the vectors,

$$\text{span}(x_1, \ldots, x_m) := \left\{ \sum_{j=1}^m \alpha_j x_j \mid \alpha_j \in \mathbb{R} \right\}.$$

(a) Show that $x_1, \ldots, x_m$ are linearly dependent if and only if there exist $\alpha_1, \ldots, \alpha_m$, not all equal to zero, such that

$$\sum_{i=1}^m \alpha_i x_i = 0.$$

(b) Suppose that the vectors $a_1, \ldots, a_n \in \mathbb{R}^n$ are linearly independent and that for two non-zero vectors $x, y \in \mathbb{R}^n$, we have

$$\sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i y_i.$$

Show that $x = y$.

(c) Show that $\{x_1, \ldots, x_m\} \in \mathbb{R}^n$ are linearly independent if and only if

$$x_i \not\in \text{span}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

for each i.

Question S.2: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if for any two vectors $x, y \in \mathbb{R}^n$, we have $f(x + y) = f(x) + f(y)$, and for any scalar $\alpha \in \mathbb{R}$ we have $f(\alpha x) = \alpha f(x)$. Let f be a linear function.

(a) Argue that for each standard basis vector

$$e_i = [\underbrace{0\ 0\ \cdots\ 0}_{i-1 \text{ times}} 1\ \underbrace{0\ \cdots\ 0}_{n-i \text{ times}}]^\top,$$

there is a vector $a_i \in \mathbb{R}^m$ such that $a_i = f(e_i)$.

(b) Argue that if $x \in \mathbb{R}^n$ with coordinates $x = [x_j]_{j=1}^n$, then

$$f(x) = \sum_{j=1}^{m} a_j x_j.$$

(c) Show that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function if and only if there exists a matrix $A \in \mathbb{R}^{m \times n}$ such that $f(x) = Ax$. What are the columns of A?

Question S.3: Consider solving the linear equation

$$Ax = b$$

where $A \in \mathbb{R}^{n \times n}$ is full rank (i.e., its n columns are linearly independent), and $x, b \in \mathbb{R}^n$. Thus, we wish to solve n equations in the n unknowns in x. Define the solution mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(b)$ is the solution to $Ax = b$.

(a) Show that (assuming $S(b)$ exists) it is unique.

(b) Show that the mapping S is linear, that is, $S(b)$ is linear in b.

(c) Conclude that there must be a matrix $B \in \mathbb{R}^{n \times n}$ such that $BA = I$, the $n \times n$ identity, where the ith column of B is $S(e_i)$. This matrix is the inverse A^{-1} of A.

Question S.4: A collection of vectors $\{u_1, \ldots, u_m\} \subset \mathbb{R}^n$ is *orthogonal* if

$$u_i \cdot u_j = u_i^\top u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in \mathbb{R}^n$. Show that the *projection* of x onto $\text{span}(u_1, \ldots, u_m)$, that is, the point $\pi(x) \in \text{span}(u_1, \ldots, u_m)$ closest to x, is

$$\pi(x) = \sum_{i=1}^{m} u_i u_i^\top x.$$

Draw a picture.